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Bianchi Type V Perfect-Fluid Space-Times

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A method is presented to generate exact solutions of the Einstein field equations in Bianchi type V space-times. The energy-momentum tensor is of perfect fluid type. Starting from particular solutions, new classes of solutions are obtained. The geometrical and physical properties of a class of solutions are discussed.

1. INTRODUCTION

Cosmological models which are spatially homogeneous and anisotropic have a significant role in the description of the universe in the early stages of its evolution. Various Bianchi spaces are useful in constructing models of spatially homogeneous cosmologies. A spatially homogeneous Bianchi model necessarily has a three-dimensional group, which acts simply transitively on a spacelike three-dimensional orbit. There is a large literature concerning specific Bianchi spaces which contain fluids with a specific equation of state. Hajj-Boutros and Sfeila (1987) have shown that the condition of the isotropy of pressure in the case of Bianchi type I space-times filled with a perfect fluid reduces via a suitable scale transformation to a linear second-order differential equation, which admits FRW models as particular solutions. They also proved the utility of the new technique to generate exact solutions with Bianchi type I symmetry with the energy-momentum tensor of a perfect fluid. Since Bianchi type I models are a very special subset of spatially homogeneous models, I here consider the more general Bianchi type V models in a similar investigation. The importance of Bianchi type V space-times is due to the fact that the space of constant negative curvature is contained in this type as a special case. Recently, Banerjee and Sanyal (1988) studied the irrotational Bianchi type V cosmological model under the influence of both shear and bulk viscosity together with heat flux.

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In this paper the Einstein field equations in the case of Bianchi type V space-times filled with a perfect fluid are considered. The condition of the isotropy of pressure leads to a nonlinear second-order differential equation which admits a class of particular solutions. A new technique is presented to generate exact solutions of the field equations. Starting from particular solutions and using the technique, new classes of solutions are obtained. The Einstein–de Sitter dust-filled model is shown as the special case of a solution. The physical and geometrical properties of some of the solutions are discussed.

2. FIELD EQUATIONS

The metric for the spatially homogeneous Bianchi type V space-time is taken in the form

$$ds^2 = dt^2 - A^2\,dx^2 - e^{2qx}(B^2\,dy^2 + C^2\,dz^2) \tag{1}$$

where A, B, C are cosmic scale functions and q is a constant. The energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\rho + p)v_\mu v_\nu - g_{\mu\nu}p \tag{2}$$

Here p is the pressure, ρ is the energy density, and v^μ is the 4-velocity, so that $v^\mu v_\mu = 1$. In the system of units where $8\pi G = C = 1$, the nonvanishing components of the Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -T_{\mu\nu} \tag{3}$$

in the comoving coordinate system $v^\mu = \delta^\mu_\nu$ are

$$\frac{B''}{B} + \frac{C''}{C} + \frac{B'C'}{BC} - \frac{q^2}{A^2} = -p \tag{4}$$

$$\frac{A''}{A} + \frac{C''}{C} + \frac{A'C'}{AC} - \frac{q^2}{A^2} = -p \tag{5}$$

$$\frac{A''}{A} + \frac{B''}{B} + \frac{A'B'}{AB} - \frac{q^2}{A^2} = -p \tag{6}$$

$$\frac{A'B'}{AB} + \frac{A'C'}{AC} + \frac{B'C'}{BC} - \frac{3q^2}{A^2} = \rho \tag{7}$$

$$\frac{2A'}{A} - \frac{B'}{B} - \frac{C'}{C} = 0 \tag{8}$$

where the prime denotes differentiation with respect to t.

Eliminating p from (4)-(6) and using (8), we find for the condition of isotropy of pressure

$$\frac{2B''}{B}+\left(\frac{B'}{B}\right)^2=\frac{2C''}{C}+\left(\frac{C'}{C}\right)^2 \tag{9}$$

The nonlinear second-order differential equation (9) admits as a particular solution

$$B(t)=C(t)=a(t) \tag{10}$$

In this case the metric (1) becomes

$$ds^2=dt^2-A^2[dx^2+e^{2qx}(dy^2+dz^2)] \tag{11}$$

It is clear that when $B(t)=C(t)=Q$, where Q is a constant, is also a particular solution of (9), and in this case (1) can be transformed into the form

$$ds^2=dt^2-dx^2-e^{2qx}(dy^2+dz^2) \tag{12}$$

If we seek a solution of (9) of the form

$$B=t^m, \qquad C=t^n \tag{13}$$

Then either $m=n$ or $m+n=2/3$. In the former case the metric (1) takes the form

$$ds^2=dt^2-t^{2m}[dx^2+e^{2qx}(dy^2+dz^2)] \tag{14}$$

In the latter case the metric of the solution can be written as

$$ds^2=dt^2-t^{2/3}\,dx^2-e^{2qx}(t^{2m}\,dy^2+t^{2n}\,dz^2) \tag{15}$$

For the metric (15), the pressure and energy-density are given by

$$p=\frac{2+9mn}{gt^2}+\frac{q^2}{t^{2/3}} \tag{16}$$

$$\rho=\frac{2+9mn}{gt^2}-\frac{3q^2}{t^{2/3}} \tag{17}$$

3. GENERATING TECHNIQUE

We now seek solutions of (9) of the form

$$B(t)=a(t)B_1(t) \tag{18}$$

or

$$C(t)=a(t)C_1(t) \tag{19}$$

Then we obtain successively

$$B(t)=a(t)\exp\left(\frac{2}{3}\left\{\int dt\left[a^3(t)\left(\int\frac{dt}{a^3(t)}+k_1\right)\right]^{-1}+k_2\right\}\right) \tag{20}$$

$$C(t)=a(t)\exp\left(\frac{2}{3}\left\{\int dt\left[a^3(t)\left(\int\frac{dt}{a^3(t)}+k_3\right)\right]^{-1}+k_4\right\}\right) \tag{21}$$

where the k's are constants of integration. Obviously $B(t)$ and $C(t)$ obtained from (20) and (21) are different from $a(t)$. Starting from particular solutions of the field equations and using (20) and (21), we now generate some new exact solutions.

To apply (20) and (21), we first consider the metric (11) where $B(t)=C(t)=a(t)=1$. Applying (20), we obtain

$$B(t)=(t+\alpha)^{2/3} \tag{22}$$

where α is an integration constant, and $C(t)=1$ stays invariable. Inserting the new variables $B(t)$ and $C(t)$ in (8) and integrating, we obtain

$$A=\mu(t+\alpha)^{1/3} \tag{23}$$

μ is an integration constant. By a change of scale the metric of the solution reads

$$ds^2=dt^2-t^{2/3}\,dx^2-e^{2qx}(t^{4/3}\,dy^2+dz^2) \tag{24}$$

For this solution the pressure and energy-density are given by

$$p=\frac{2}{9t^2}+\frac{q^2}{t^{2/3}} \tag{25}$$

$$=\frac{2}{9t^2}-\frac{3q^2}{t^{2/3}} \tag{26}$$

If we apply both (20) and (21), we obtain

$$B=(t+\alpha)^{2/3} \tag{27}$$

and

$$C=(t+\beta)^{2/3} \tag{28}$$

α, β are integration constants. Inserting (27) and (28) into (8) and integrating, we find that

$$A = \nu(t+\alpha)^{1/3}(t+\beta)^{1/3} \tag{29}$$

ν is an integration constant. The metric of the solution can be written in the form

$$ds^2 = dt^2 - (t+\alpha)^{2/3}(t+\beta)^{2/3}\, dx^2 - e^{2qx}[(t+\alpha)^{4/3}\, dy^2 + (t+\beta)^{4/3}\, dz^2] \tag{30}$$

The pressure and energy-density are given by

$$p = \frac{2(\alpha-\beta)^2}{9t^2(t+\alpha)^2(t+\beta)^2} + \frac{q^2}{(t+\alpha)^{2/3}(t+\beta)^{2/3}} \tag{31}$$

$$\rho = \frac{12t^2 + 12(\alpha+\beta)t + 2(\alpha^2+\beta^2+4\alpha\beta)}{9(t+\alpha)^2(t+\beta)^2} - \frac{3q^2}{(t+\alpha)^{2/3}(t+\beta)^{2/3}} \tag{32}$$

For $\alpha=\beta$, $q=0$ we obtain the Einstein and de Sitter (1932) dust-filled model. The pressure and energy-density tend to zero as $t\to\infty$; the model would give at large time an essentially empty universe. The expansion and shear scalar are (Ellis, 1971)

$$\theta = \frac{2t+\alpha+\beta}{(t+\alpha)(t+\beta)} \tag{33}$$

$$\sigma = \frac{[7t^2 + 7(\alpha+\beta)t + 2(\alpha^2+\beta^2) + 3\alpha\beta]^{1/2}}{3(t+\alpha)(t+\beta)} \tag{34}$$

The shear scalar σ is finite for $0<t<\infty$ and tends to zero as $t\to\infty$; the universe is then shear-free, and there is no anisotropy. It is also seen that the ratio σ/θ does not tend to zero as $t\to\infty$, which means that σ does not tend to zero faster than the expansion. For the model all of the fluids are acceleration- and rotation-free.

Applying (20) and (21) to the metric (30), we arrive at the same solution with different parameter.

If we apply (20) and (21) to (14), we can write the metric of the corresponding solution in the form

$$\begin{aligned} ds^2 = dt^2 &- t^{-2m}(t+c_1t^{3m})^{2/3}(t+c_2t^{3m})^{2/3}\, dx^2 \\ &- e^{2qx}[t^{-2m}(t+c_1t^{3m})^{4/3}\, dy^2 + t^{-2m}(t+c_2t^{3m})^{4/3}\, dz^2] \end{aligned} \tag{35}$$

For isotropy of pressure $c_1=c_2=c$, the metric (35) reduces to

$$ds^2 = dt^2 - t^{-2m}(t+ct^{3m})^{4/3}[dx^2 + e^{2qx}(dy^2+dz^2)] \tag{36}$$

The pressure and energy-density are given by

$$p = -\frac{3m^2+2m}{t^2} - \frac{4cm(3m-1)t^{3m-2}}{t+ct^{3m}} + \frac{4m}{t}\frac{1+3cmt^{3m-1}}{t+ct^{3m}} + \frac{q^2}{t^{-2m}(t+ct^{3m})^{4/3}} \tag{37}$$

$$\rho = \frac{3m^2}{t^2} + \frac{4}{3}\left(\frac{1+3cmt^{3m-1}}{t+ct^{3m}}\right)^2 - \frac{4m}{t}\frac{1+3cmt^{3m-1}}{t+ct^{3m}} - \frac{3q^2}{t^{-2m}(t+ct^{3m})^{4/3}} \tag{38}$$

4. CONCLUSION

A new method is presented for generating exact solutions of Bianchi type V models filled with a perfect fluid. A class of expanding and anisotropic cosmological models is obtained in which all of the fluids are acceleration- and rotation-free. These models can be added to the rare perfect fluid solutions of this type not satisfying the equation of state.

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